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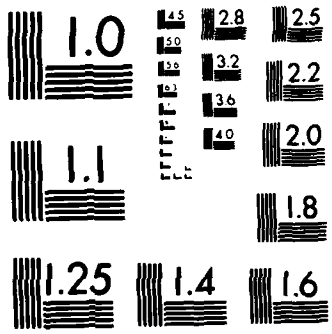

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ASYMPTOTIC CONSTRUCTION OF A PROCEDURE FOR PLANE-WAVE SYNTHESIS AND MIGRATION

by

Robert D. Mager

September, 1984

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Colorado School of Mines
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Department of Mathematics
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1. INTRODUCTION

Several papers in recent years [1,2,3] have dealt with the construction and migration of plane-wave seismograms. A plane-wave seismogram can be constructed by performing a slant-stack over source or receiver coordinates with a constant step-out in time per trace. Based on travel-time considerations, it is then possible to conceive of an imaging procedure based on the concept of downward continuation. This process images structure by propagating wavefields in reverse back to the reflectors from which they originated.

It is not possible, from travel-time considerations only, to arrive at the correct amplitude or weighting factors to be used in the imaging process. Ideally, the amplitudes on a migrated or imaged section should be proportional only to the local reflection strength. The imaged section should be free from source-receiver and geometrical spreading factors, and also free from focusing effects due to the curvature of the reflecting surface.

In a recent paper in Geophysics [3], Trietel, et al, examine the question of amplitude dependence on plane-wave seismograms. Their work was based on a plane-wave decomposition technique described by Müller[4]. The analysis contained in the Trietel paper is strictly valid only for horizontal reflectors located in a vertically stratified medium. Their numerical examples, however, show that the method has a much wider range of applicability to more complex geometries.

This report outlines the construction of a more exact operator for the migration of plane-wave seismograms. The objective here is to construct a procedure which will produce an output appropriate for both structural imaging and velocity inversion; that is, an output which is a direct indicator of reflection strength. The calculations presented here allow the reflector to have an arbitrary complex two-dimensional shape. While the reflector is assumed to be two-dimensional, the propagating wavefields are assumed to be three dimensional and are responses from point sources. The analysis presented here deals with the case of a constant background velocity field. More general cases are now being examined.

2. SUMMARY OF THE ANALYSIS

Figure (1) shows the geometrical quantities involved in this discussion. A source of acoustical waves is located on the surface at $\underline{X}_s = (x_s,0,0)$, giving rise to an incident field which is scattered from the reflection interface Σ. These scattered waves are then detected on the surface by a series of receivers located at $\underline{X}_g = (x_g,0,0)$. The scattered observations are denoted by $P_S(x_s,x_g,t)$.

By an elementary application of Huygen's principle, an incident plane wave front may be synthesized by use of an array of spherical sources along the surface which are separated in time by an appropriate delay. This generates a sequence of point source responses which are then summed to synthesize a plane wave front. In the time domain, this summation takes the form

$$P_{PW}(x_g,\alpha,t) = \sum_s P_S\left[x_g,x_s,\ t - (\Delta s\ \cos\alpha)/v\right] \tag{2.1}$$

where v is the medium velocity and α is the angle of the plane wave with the horizontal. The symbol Δs refers to the spacing between sources and the factor $(\Delta s \cos\alpha)/v$ gives the appropriate time delay between the successive source initiation times. The quantity $v/\cos\alpha$ is thus the horizontal velocity of the plane wave and $(\Delta s \cos\alpha)/v$ is the time required for this wave front to travel between two separate source locations.

The Fourier-transform equivalent of (2.1) is

$$P_{PW}(x_g,\alpha,\omega) = \sum_s P_S \left[x_g, x_s, \omega \right] e^{-i(\omega/v)\Delta s \cos\alpha} \qquad (2.2)$$

That the summation in (2.2) produces a plane-wave seismogram may be seen by regarding the summation as an integral over the s variable and evaluating that integral by one-dimensional stationary phase. It is useful here, however, to formulate a more comprehensive approach in which the plane-wave summation and the migration summation are considered in unison. All resulting integrals can then be evaluated asymptotically by the method of multi-dimensional stationary phase.

We thus construct an image or migrated field in the form

$$P_{IM}(\underline{X}_D) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dx_s \int_{-\infty}^{\infty} dx_g \, A_M(x_s,x_g,\alpha,s)\, P_S(x_s,x_g,\omega) \qquad (2.3)$$

This formula involves an integral over receivers followed by an integral over sources. The former constructs the equivalent here of downward continuation of the upward scattered wavefronts, whereas the later provides the plane-wave construction. The integration over ω constitutes the usual imaging step of migration, where the zero-time data at each depth level or image point constructs the final migrated image. The vector $\underline{X}_D = (x_D, 0, z_D)$ represents the point of downward continuation where the output image is evaluated.

From travel time considerations, the operator A_M is taken to have the form

$$A_M = A(k,\alpha)\ e^{-ik\left[(x_D - x_s)\cos\alpha\ + z_D \sin\alpha + r_D\right]} \qquad (2.4)$$

with the wavenumber $k = \omega/v$.

Here, r_D represents distance along the path of downward continuation. The structure of the phase in this expression is based on some hindsight, and is constructed in order to cancel the phase in the point source responses.

The stationarity conditions to be described below will show that the exponential field of (2.3) represents the total phase of a plane-wave at angle α, with the line of zero phase passing through the source point. The $x_s \cos\alpha$ term in the exponential argument gives the appropriate delay across sources for the plane-wave construction.

The result of the stationary phase analysis, the details of which are outlined in the next section, is that the amplitude factor $A(k,\alpha)$ is chosen as

$$A(k,\alpha) = -\ \frac{4\pi|k|z_D}{r_D}\sqrt{\frac{r_s + r_D}{r_D}}\ \sin\alpha \qquad (2.4)$$

The geometrical quantities involved here are shown in Figure (2). The geometrical distance r_s at first appears to be unknown since it is distance from some source location to a specular reflection point. Hence, it would appear to depend on a priori knowledge of the reflector. However, the

stationarity conditions illustrated in Figure (3) show that r_s may be replaced by

$$r_s = (x_D - x_s)\cos\alpha + z_D \sin\alpha \tag{2.6}$$

This gives the correct geometrical equivalent for r_s when the image point $\underline{x}_D$ coincides with the specular point $\underline{x}'$.

The above choice of migration amplitude gives an output image that has a particularly simple form. Setting

$$r_g = r_D + \varepsilon \quad ,$$

then as $r_D \to r_g$, $\varepsilon \to 0$. Evaluation of the spatial integrals in (2.2) by stationary phase then gives

$$P_{IM}(\underline{x}_D, k) = 2\pi R\ S(\omega)\ e^{i(\omega/v)\varepsilon} + O(\varepsilon) \tag{2.7}$$

Here $S(\omega)$ is the Fourier transform of the source wavelet, and R is the geometrical optics reflection coefficient. The integral over ω in (2.2) can be calculated exactly, giving

$$P_{IM}(\underline{x}_D) = R\ S\ (\varepsilon/v) \tag{2.8}$$

This last equation is the key result of this analysis, and illustrates the mechanism by which the reflector image is created. Across any zone of specular reflection, as shown in Figure (4), a family of functions having the form of (2.7) are traced across the surface, with ε representing normal distance to the reflector. The migrated output is thus directly proportional to the local geometrical optics reflection coefficient and to the area under the Fourier transform of the wavelet.

From the geometrical optics reflection coefficient, it is possible to infer some information concerning the jump in velocity across Σ. For plane waves of incident angle θ_I,

$$R = \frac{\cos\theta_I - \sqrt{\cos^2\theta_I - \delta_v}}{\cos\theta_I + \sqrt{\cos^2\theta_I - \delta_v}} \tag{2.9}$$

Here δ_v is the relative variation across Σ of the square of the propagation velocity:

$$\delta_v = \frac{v_o^2 - v^2}{v^2} \tag{2.10}$$

Equation (2.8) can be solved for the variation δ_v giving

$$\delta_v = \frac{4\cos^2\theta_I R}{(1+R)^2} = \frac{4\sin^2[(\alpha+\theta_g)/2]R}{(1+R)^2} \quad . \tag{2.11}$$

It is then straightforward to calculate v in terms of v_o and δ_v by using (2.10). The angle θ_I, however, can only be inferred from a knowledge of the reflector tilt, which must be determined from the imaged section.

3. The Stationary Phase Analysis

In Appendix A, the construction of an integral formula for the reflection response at $\underline{x}_g$ due to a point source at $\underline{x}_s$ is outlined. That analysis is based on the use of a Green's identity to construct an integral expression for the field $P_S(\underline{x}_g,\underline{x}_s,\omega)$. A Kirchhoff approximation is then assumed for the unknown fields appearing in the integrand of that result. The integral over the surface in the direction of symmetry of the reflector is eliminated by one-dimensional stationary phase, leading to the expression.

$$P_S(x_g,x_s,k) = \frac{|k|^{1/2}}{4(2\pi)^{3/2}} e^{-i(\pi/4)\,\mathrm{sgn}(k)} S(\omega) \cdot \int_{\Sigma'} R(\sigma) \frac{\hat{N}\cdot(\hat{r}_s+\hat{r}_g)}{\sqrt{r_s r_g (r_s + r_g)}} e^{ik(r_s+r_g)} d\sigma \tag{3.1}$$

The integral here is an integral in arc-length σ across a typical cross-sectional profile Σ' of the reflection. A combination of (3.1) with (2.2) and (2.3) leads to a three-dimensional integral in the variables (x_s,x_g,σ), followed by the integration over ω:

$$P_{IM}(\underline{x}_D) = \int_{-\infty}^{\infty} d\omega\ S'(\omega) \int_{-\infty}^{\infty} dx_s \int_{-\infty}^{\infty} dx_g \int_{-\infty}^{\infty} \frac{\hat{N}\cdot(\hat{r}_s + \hat{r}_g)}{\sqrt{r_s r_g (r_s + r_g)}}\, e^{ik\Phi(x_g,x_s,\sigma)}\, d\sigma \tag{3.2}$$

where

$$S'(\omega) = \frac{|k|^{1/2}\, e^{i(\pi/4)\,\mathrm{sgn}(k)}}{4(2\pi)^{3/2}}\ S(\omega)$$

and the phase function $\Phi(x_g, x_s, \sigma)$ is

$$\Phi(x_g, x_s, \sigma) = r_s + r_g - (x_D - x_s)\cos\alpha - z_D \sin\alpha - r_D \quad . \tag{3.3}$$

The explicit dependence of this function on the variables x_g, x_s and σ is given through the distance functions

$$r_s = \left|\underline{x}_s - \underline{x}'(\sigma)\right| \qquad r_g = \left|\underline{x}_g - \underline{x}'(\sigma)\right|,\ r_D = \left|\underline{x}_D - \underline{x}_g\right| \quad ,$$

where $\underline{x}'(\sigma)$ is a point of integration on the scattering surface.

The conditions for stationarity in (x_g, x_s, σ) of this phase function are

$$\Phi_\sigma = \frac{\partial}{\partial\sigma}(r_s + r_g) = \hat{T}\cdot(\hat{r}_s + \hat{r}_g) = 0$$

$$\Phi_s = \frac{\partial}{\partial x_s}(r_s + x_s \cos\alpha) = -\cos\Theta_s + \cos\alpha = 0 \qquad (3.4)$$

$$\Phi_g = \frac{\partial}{\partial x_g}(r_g - r_D) = \cos\Theta_g - \cos\Theta_D = 0$$

Figures (2) and (3) show the geometrical quantities before and after the conditions for stationarity are imposed. The first of these is a requirement for specular reflections (The symbol $\hat{T}$ refers to the unit tangent vector along Σ'.) The second condition specifies that the ray angle (with the horizontal) at the source point must equal the plane-wave angle α. The last equation is a statement that the downward continuation angle Θ_D equals the receiver ray angle Θ_g.

Mathematically, these equations are specifying geometrical configurations where the dominant asymptotic contributions to the three-fold spatial integral in (2.2) occur. Physically, the stationary directions are the directions of spatial coherency. In the directions specified by these equations, wavefronts combine constructively in phase to give a leading-order contribution. In other directions, wavefronts combine destructively to give a lower-order contribution. It is this constructive and destructive interference that provides the creation of (1) a net incident wavefront which is planar, and (2) a backward extrapolation of the scattered waves which is in exact reverse to the directions of forward propagation.

The stationary phase formula, applied to a three-fold integral of the form

$$I(\lambda) = \int\int_{-\infty}^{\infty}\int f(\underline{x})\, e^{i\lambda g(\underline{x})} d\underline{x} \quad \underline{x}=(x_1,x_2,x_3)$$

is

$$I(\lambda) \sim \sum_j \left[\frac{2\pi}{|\lambda|}\right]^{3/2} f(\underline{x}_j) \frac{e^{i\left[\lambda g(\underline{x}_j) + \pi/4 \ (\operatorname{sgn} \lambda) \ \operatorname{sig}(W_j)\right]}}{\left|\det(W_j)\right|} \tag{3.5}$$

Here W_j is the matrix of second derivatives

$$W_j = \left(\frac{\partial^2 g}{\partial x_m \, \partial x_n}\right) \qquad m, n = 1,2,3 \quad .$$

Also, $\det(W)$ is the determinant of this matrix and $\operatorname{sig}(W_j)$ is its signature. The summation is over all possible stationary points and the subscript on W_j indicates an evaluation at $\underline{x}_j$.

For the problem at hand, the computation of the quantities $\det(W_j)$ and $\operatorname{sig}(W_j)$ can be simplified since, by some hindsight, we anticipate that these quantities will be of interest where

$$r_s \rightarrow r_g \quad .$$

That is, we are interested in the behavior of the spatial integrals as the image point $\underline{x}_D$ of downward continuation approaches a specular point $\underline{x}'$ on

the reflector. The details of this calculation are given in Appendix B. There, it is shown that

$$|\det(W)| = \frac{z_D \sin\alpha \cos\theta_I}{r_g^2 \, r_s^{1/2}} \tag{3.6}$$

and

$$\mathrm{Sig}(W) = +1 \quad .$$

The factor of $\cos\theta_I$ provided by the presence of $|\det(W)|$ in the denominator cancels (except for a factor of 2) with $\hat{N}\cdot(\hat{r}+\hat{r}_g)$ in the numerator of (3.2), since, at stationary,

$$\hat{N}\cdot(\hat{r}_s+\hat{r}_g) = 2\cos\theta_I \quad .$$

Also, the phase delay of $\pi/4$ exactly cancels with the phase $\pi/4$ in (2.1). The net result of the stationary phase analysis is that

$$P_{IM}(\underline{x}_D) = \int_{-\infty}^{\infty} \frac{A(k,\alpha)R}{2(k)\ \sin\alpha} \left[\frac{r_g}{z_D}\right] \sqrt{\frac{r_g}{r_s+r_g}} \; e^{ik\,\Phi}\, S(\omega)\, d\omega \ . \tag{3.7}$$

The explicit dependence of these variables is not indicated; however, it is understood that all quantities satisfy the conditions for stationarity and that (3.7) strictly applies in the neighborhood of an image point. Also, the summation over all possible stationary points is understood.

The similarity of (3.7) to a Fourier transform suggests that the amplitude factor $A(k,\alpha)$ should be selected so as to cancel the unwanted geometrical factors in that equation. Since our interest is in the neighborhood of an image point on the target reflector, we set $r_g = r_D$ and

$$\begin{aligned} A(k,\alpha) &= 4\pi \sin\alpha \left[\frac{z_D}{r_g}\right] \sqrt{\frac{r_s+r_g}{r_D}} \\ &= 4\pi \sin\alpha \, \sin\theta_D \sqrt{\frac{r_s+r_D}{r_D}} \end{aligned} \tag{3.8}$$

Also, Figure (3) shows that we may replace r_s in this formula by

$$r_s = (x_D - x_s)\cos\alpha + z_D \operatorname{Sin}\alpha$$

This choice for $A(k,\alpha)$ leads to the results suggested in the previous section.

In summary, the direction of this analysis has been to assume a certain canonical form for an imaging operator. This canonical form is not specific except for certain identified phase factors which provide a reverse extrapolation of scattered wavefields. By stationary phase analysis the amplitude of the image field can be calculated, and this leads to a selection of the correct amplitude for the migration operator. This process is inversion in the sense that the emphasis is placed upon achieving a final image which depends only upon a velocity contrast or some reflection coefficient. However, it is an indirect process in that it does not involve

an analytical solution of some scattering equation.

The philosophy of the analysis involved here should have useful applications for other types of exploration problems. Specifically, it may be useful for constructing migration operators which provide specified cross-sectional images from three-dimensional data volumes. These operators can be designed with more consideration given to the construction of true amplitude results. This approach does not require the migration operator to be a backward solution of the wave equation. It allows instead for a more general formulation of the imaging and inversion problem.

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APPENDIX A

Construction of the Kirchhoff Integral Representation

We describe here the details of the analysis leading to equation (3.1), which gives the scattered or upgoing response as a surface integral over the reflector.

We define first a total acoustic field which is the sum of the incident and scattered fields. It is given as a solution to the equation.

$$(\nabla^2+k^2)P = -\,\delta(\underline{x}-\underline{x}_s)\,S(\omega), \quad k = \omega/v \quad . \tag{A.1}$$

The field $P(\underline{x},\underline{x}_s,k)$ and its normal derivative are required to be continuous across the reflector, and $P(\underline{x},\underline{x}_s,k)$ is also required to satisfy a radiation condition at ∞:

$$\lim_{R\to\infty} R\left[\frac{\partial P}{\partial R} + ik\,P\right] = 0 \quad , \quad R = |\underline{x}| \; .$$

The scattered field $P_S(\underline{x},\underline{x}_s,k)$ is defined by the equation

$$P = P_I + P_S \quad ,$$

where

$$P_I(\underline{x},\underline{x}_s,k) = \frac{e^{ikR_s}}{4\pi R_s}\,S(\omega) \quad , \quad R_s = |\underline{x}-\underline{x}_s| \; . \tag{A.2}$$

The function P_I is the incident wavefield, and represents the free-space response to a point source at $\underline{x}_s$. By a standard Green's function argument, P_S can be written as an integral over the surface of the reflector:

$$P_S(\underline{x}_g,\underline{x}_s,k) = \int_\Sigma\int \left[P_s(\underline{x}',\underline{x}_s,k)\,\frac{\partial G(\underline{x}';\underline{x}_g)}{\partial n'} - G(\underline{x}';\underline{x}_g)\,\frac{\partial P_s(\underline{x}',\underline{x}_s,k)}{\partial n'}\right] d\Sigma \tag{A.3}$$

The free-space Green's function used here is

$$G(\underline{x}';\underline{x}_g) = \frac{e^{ikR_g}}{4\pi R_g} \quad , \quad R_g = \left|\underline{x}'-\underline{x}_g\right| \tag{A.4}$$

and $\hat{n}'$ is the upward normal from the reflector.

We now invoke a Kirchhoff approximation in order to simplify the integrand of (A.2). An "illuminated" region L on the surface Σ is defined as that portion of the surface intersected by straight line rays from the source. The remainder of the surface is denoted by D. We make the approximation

$$
\left.
\begin{aligned}
P_S(\underline{x},\underline{x}_s,k) &= R\, P_I(\underline{x},\underline{x}_s,k) \\
&\qquad\qquad \text{on } L \\
\frac{\partial P_S(\underline{x},\underline{x}_s,k)}{\partial n'} &= -\, R\, \frac{\partial P_I}{\partial n'}(\underline{x},\underline{x}_s,k)
\end{aligned}
\right] \tag{A.5}
$$

$$
\frac{\partial P_s}{\partial n'}(\underline{x},\underline{x}_s,k) = P(\underline{x},\underline{x}_s,k) = 0 \qquad \text{on } D \quad .
$$

These are the relationships predicted by geometrical optics. The reflection coefficient R will be taken as the plane-wave reflection coefficient, which for waves of incident angle θ_I is given by (2.8). A more detailed description of the Kirchhoff approximation is given by Bleistein [5].

The combination of (A.2) - (A.5) leads to

$$
P_S(\underline{x}_g,\underline{x}_s,k) = \frac{S(\omega)}{(4\pi)^2}\, R \int_L\!\!\int \frac{\partial}{\partial n'} \frac{e^{ik(R_s+R_g)}}{R_s R_g}\, d\Sigma \quad .
$$

We eliminate the component of integration in the direction y' of symmetry by one-dimensional stationary phase [5]. The result is

$$P_S(\underline{x}_g,\underline{x}_s;k) = \frac{S(\omega)}{8\pi(2\pi)^{1/2}} R\, e^{i(\pi/4)\mathrm{sgn}(k)} |k|^{-1/2} \int_{L_r} \frac{\partial}{\partial n'} \left[\frac{e^{ik(r_s+r_g)}}{\sqrt{r_s r_g}\,\sqrt{r_s+r_g}} \right] d\sigma' \quad (A.6)$$

The final step in the simplification of (A.6) is the calculation of the normal derivative of the integrand according to

$$\frac{\partial}{\partial n'} = \nabla' \cdot \hat{n}'$$

Only the term arising from differentiation of the phase in the exponential is kept, since terms arising from derivatives of the amplitude are of lower order in k. A further precaution when simplifying (A.6) is to note that

$$\mathrm{Sgn}(k) = \begin{cases} -1 & k < 0 \\ +1 & k > 0 \end{cases} \quad (A.7)$$

This arises because a minus sign has been used in the definition of the forward Fourier transform. We use these results in the stationary phase formula (3.5) and obtain (3.1).

APPENDIX B

"Calculation of the Matrix Determinant and Signature"

For application of formula (3.5), the determinant and signature of the matrix of second derivatives must be determined. These quantities are needed whenever the conditions for stationarity are met, hence equations (3.4) are applied in these calculations. This leads to the following quantities:

$$\Phi_{\sigma\sigma} = \left(\frac{1}{r_s} + \frac{1}{r_g}\right) \cos^2 \theta_I - 2\eta K \cos \theta_I$$

$$\Phi_{ss} = \frac{\sin^2\theta_s}{r_s} = \frac{\sin^2\alpha}{r_s}$$

$$\Phi_{gg} = \frac{\sin^2\theta_g}{r_g} - \frac{\sin^2\theta_D}{r_D}$$

$$\Phi_{s\sigma} = - \frac{-\cos\theta_T}{r_s} + \frac{\cos\theta_s \sin\theta_I}{r_s}$$

$$\Phi_{g\sigma} = \frac{-\cos\theta_T}{r_g} + \frac{\cos\theta_g \sin\theta_R}{r_g}$$

$$\Phi_{sg} = 0 \quad .$$

In the *first* of these equations $\underline{K}$ is the curvature vector of Σ' at the stationary point; K is its magnitude and $\eta = -\operatorname{sgn} \underline{K}\cdot\hat{n}$, which is +1 for an anticlinal curve and −1 for a synclinal curve.

In addition, these quantities are needed as $r_D \to r_g$; i.e., near a specular image point on the reflector. Hence set $r_D = r_g$ and arrive at the following expression for the matrix of second derivatives.

$$W = \begin{bmatrix} \dfrac{\sin^2\alpha}{r_s} & 0 & \dfrac{-\cos\theta_I \sin\alpha}{r_s} \\[2ex] 0 & 0 & \dfrac{-\cos\theta_I \sin\theta_D}{r_g} \\[2ex] \dfrac{-\cos\theta_I \sin\alpha}{r_s} & \dfrac{-\cos\theta_I \sin\theta_D}{r_g} & \begin{array}{c} \cos^2\theta_I \left[\dfrac{1}{r_s} + \dfrac{1}{r_g}\right] \\ -2\eta K \cos\theta_I \end{array} \end{bmatrix}$$

The determinant of this matrix is therefore

$$\det(W) = \frac{z_D^2 \sin_\alpha^2 \cos^2\theta_I}{r_g^4 \, r_s} ,$$

where $\cos\theta_D = z_D/r_D$.

The characteristic polynomial for W is

$$\lambda^3 - \lambda^2\ [a + b] - \lambda\ [-ab + c + d] + ac \quad ,$$

where

$$a = \frac{\sin^2 \alpha}{r_s}\, \alpha \quad ,$$

$$b = \cos^2\ \theta_I \left[\frac{1}{r_s} + \frac{1}{r_g}\right] - 2\eta K\ \cos\ \theta_I \quad ,$$

$$c = \frac{\cos^2 \theta_I \sin^2 \theta_D}{r_g^2} \quad .$$

The following argument is made for the determination of the signs of the roots of this characteristic equation. The quantity $-ac$ gives the product of the three roots; this quantity is always negative. Hence, either all three roots are negative, or one root is negative and the remaining two are positive. We then have the following sub-cases:

1) $\eta < 0$ (reflector concave down). In this case $a + b > 0$. But $a + b$ gives the sum of all three roots. This implies that all three roots cannot be negative. Therefore, there must be two positive roots and one negative root, which implies a signature of +1.

2) $\eta > 0$ (reflector concave up), but a + b is positive. Again, in this case, there are two positive roots and one negative root, and Sig w = + 1.

3) $\eta > 0$ (concave up), but a + b is negative. In this case, the term b must be negative, which implies that ab−c−d is negative. It can now be argued by Descartes' rule of signs that again there must be two positive roots and one negative root.

Since the above exhausts all possibilities, cases, we have that in each case:

$$\text{Sig } w = 2 - 1 = +1 \quad .$$

LIST OF FIGURE CAPTIONS

1. Figure (1): Illustrating the reflection geometry.

2. Figure (2): Illustrating geometrical quantities before stationary phase.

3. Figure (3): Illustrating geometrical quantities after stationary phase. The angle at the source is α, $\hat{r}_s$ and $\hat{r}_g$ make equal angles with the surface normall $\hat{N}$, and $\theta_g = \theta_D$.

4. Figure (4) Showing the image reconstruction of a zone of specular reflection on the target surface.

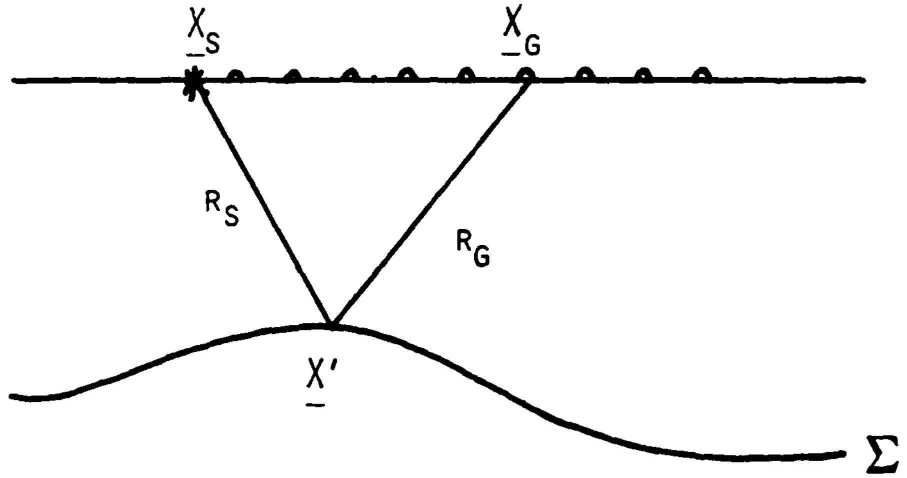


FIGURE 1

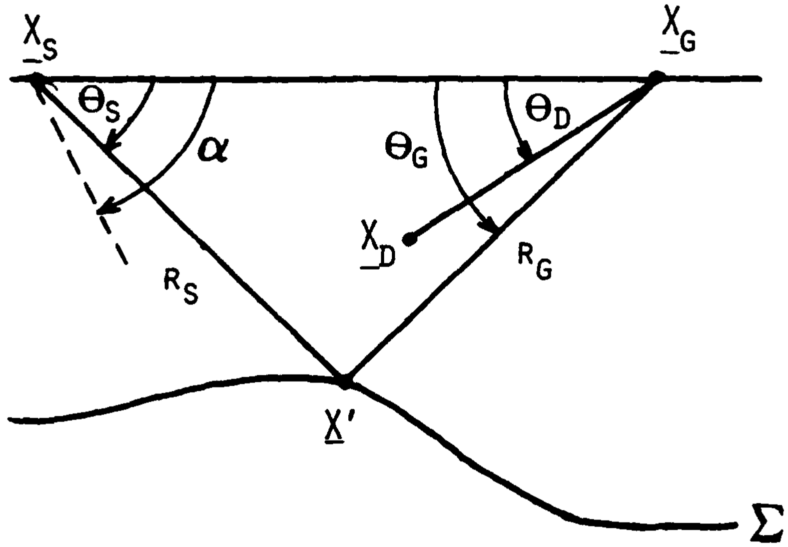
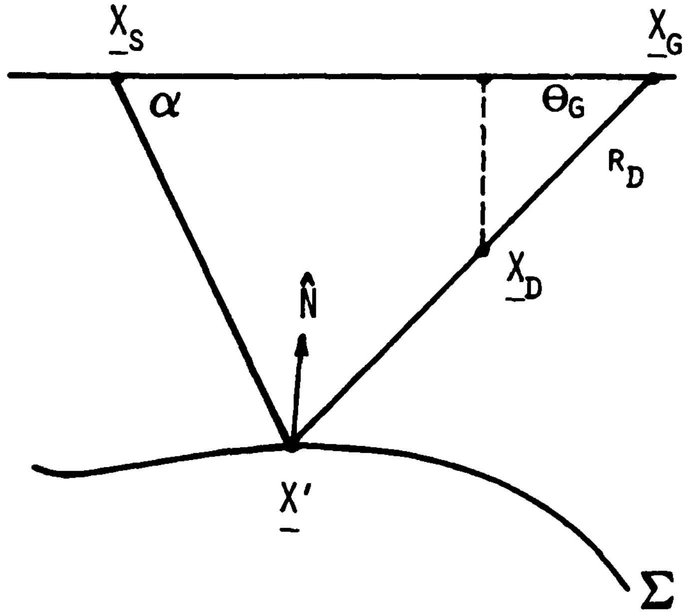


FIGURE 2



$\underline{X}_S$
$\underline{X}_G$
α
Θ_G
R_D
$\underline{X}_D$
$\hat{N}$
$\underline{X}'$
Σ

FIGURE 3

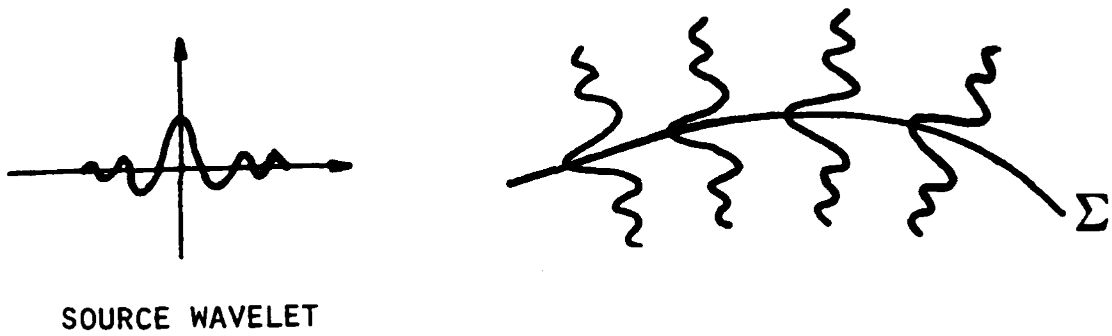


FIGURE 4

END

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